SOME NONPARAMETRIC UMP TESTS FOR SPHERICAL SYMMETRY

by

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Summary

Let $X = (X_1,\ldots,X_n)'$ be an $n \times 1$ random vector with a probability density function (pdf) f. Let $\mathfrak{F}_0$ be the class of pdf's in R^n invariant under the orthogonal group $\mathfrak{O}(n) = \{g:n \times n \mid gg' = I_n\}$. Let $\mathfrak{F}_1(\Sigma)$ be the class of the pdf's of the form $h(x) = c|\Sigma|^{-\frac{1}{2}}q(x'\Sigma^{-1}x)(x \in R^n)$ where q is a nonincreasing function on $[0, \infty)$. For certain choices of Σ, we derive UMP (uniformly most powerful) tests for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_1(\Sigma)$. A similar result for a location alternative is also given. Further the distributions of some functions of $X/\|X\|$ are considered when the distribution of X is spherically symmetric. Those results are applied to a regression model.

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§ 0. Introduction and Notation.

Let $\mathfrak{O}(n) = \{g:n\times n \mid gg' = I_n\}$ be the orthogonal matrix group. Let $\mathcal{L}(X)$ denote the distribution law of a random vector (variable) X. An $n\times 1$ random vector X is said to have a spherically symmetric distribution if $\mathcal{L}(gX) = \mathcal{L}(X)$ for all $g \in \mathfrak{O}(n)$. By $\mathcal{L}(X) \in S(n)$ we mean that a given $n\times 1$ random vector X has a spherically symmetric distribution. Let $\mathfrak{F}_0$ be the class of the pdf's (probability density functions) in R^n invariant under $\mathfrak{O}(n)$, i.e., $f \in \mathfrak{F}_0$ if and only if $f(gx) = f(x)$ for all $g \in \mathfrak{O}(n)$. Let $\mathfrak{F}_1(\Sigma)$ by the class of the pdf's of the form

$$(0.1) \qquad h(x) = c|\Sigma|^{-\frac{1}{2}} q(x'\Sigma^{-1}x) \qquad (x \in R^n) ,$$

where q is a nonincreasing function of $[0, \infty)$ and the constant c depends on q only. Here Σ is an $n\times n$ positive definite matrix and has a certain structure specified later Let $\mathfrak{F}_2(\mu)$ be the class of the pdf's of the form

$$(0.2) \qquad h(x) = cq(\|x - \mu a_0\|^2) \quad (x \in R^n),\ \mu \in R^1 ,$$

where q and c are the same as above, $a_0 \neq 0$ is an $n\times 1$ known vector and $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ for $x = (x_1,\ldots,x_n)'$. Here μ is either positive or negative. We note that (0.2) may be written as $h(x) = \frac{c}{\sigma^n} q'(\|x-\mu a_0\|^2/\sigma^2)$ but the scale parameter σ^2 can be absorbed into q. Let $X = (X_1,\ldots,X_n)'$ be an $n\times 1$ random vector with a pdf f. In this paper we consider the testing problems: $f \in \mathfrak{F}_0$ versus $f \in \mathfrak{F}_1(\Sigma)$ and $f \in \mathfrak{F}_0$ versus $f \in \mathfrak{F}_2(\mu)$.

Lehmann and Stein (1949) (abbreviated as L-S(1949) below), in connection with permutation tests, briefly treated the testing problem:

$f \in \mathfrak{F}_0$ versus the alternative that X_i's are independently identically distributed (i.i.d.) normally distributed with mean $\mu > 0$ (or $\mu \neq 0$) and variance σ^2. We note that in our problem the independence of X_i's is not assumed under either of the hypotheses. In fact, assuming the independence with spherical symmetry is equivalent to assuming normality for X. We also note that the alternatives $\mathfrak{F}_1(\Sigma)$ and $\mathfrak{F}_2(\mu)$ contain such distributions as a multivariate t-distribution (or a multivariate Cauchy distribution) etc., besides a multivariate normal distribution. The readers are referred to Johnson and Kotz (1972) for special multivariate distributions of the form (0.1) or (0.2). Some properties of spherically symmetric distributions are studied by Kelker (1970).

We summarize our work. In § 1 we consider $\mathcal{L}(y/\|y\|)$ when $\mathcal{L}(y) \in S(n)$ and $P(y = 0) = 0$. Based on the result, we express $\mathcal{L}(y'Ay/y'y)$ in terms of a Dirichlet distribution where A is a symmetric matrix. When the pdf of y exists, Kelker (1970) treated this problem. As an example, the distribution of a sample correlation coefficient is given when only one of the samples is spherically distributed. The results here are applied in §2 $\sim$ §4. In §2, for a fixed Σ, we derive a UMP test of a level α for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_1(\Sigma)$ and then apply this result when Σ has special forms: (1) $\Sigma = \sigma^2 \Sigma_0$ (Σ_0: known). (2) $\Sigma = \sigma_1(I-M) + \lambda_2 M$, $M^2 = M$, $\lambda_1 > \lambda_2 > 0$ (or $\lambda_2 > \lambda_1 > 0$) and (3) $\Sigma^{-1} = \lambda_1 I + \lambda_2 A$ (A: known), $\lambda_1 > 0$. There the UMP test does not depend on fixed unknown parameters, so it is UMP over all unknown parameters as well as the function q. We note that Σ of the form (2) contains the intra-class covariance structure and Σ of the form (3) often appears in serial correlation problems. We also calculate the power functions of the UMP tests for some cases. Similarly in §3, a UMP test for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_2(\mu)$ is derived. The approach here is similar to that in L-S(1949). In fact we modify a theorem in L-S(1949) for the hypothesis of invariance under a finite group into a theorem for the case of an infinite group. We remark that L-S themselves gave a version for an infinite group (§6) but it seems difficult to apply. In §4 the above results are applied to the problem of testing a general linear hypothesis in a regression model.

§1 The distributions of $a'X/\|X\|$ and $X'AX/X'X$.

Let $N_n(\Delta, \Sigma)$ denote an n-dimensional normal distribution with mean Δ and covariance matrix Σ. Let $D_n(a_1,\ldots,a_{n-1}:a_n)$ denote a Dirichlet distribution with pdf

$$p_n(t_1,\ldots,t_{n-1}) = \Gamma(\sum_{i=1}^{n-1} a_i)[\prod_{i=1}^{n}\Gamma(a_i)]^{-1}[\prod_{i=1}^{n-1} t_i^{a_i-1}](1 - \sum_{i=1}^{n-1} t_i)^{a_n-1}$$

where $0 \le t_i$ and $\sum_{i=1}^{n-1} t_i \le 1$. By $\mathcal{L}(y_1,\ldots,y_n) = D_n(a_1,\ldots,a_{n-1}:a_n)$ we mean that $y_n = 1 - \sum_{i=1}^{n-1} y_i$ and $(y_1,\ldots,y_{n-1})$ has the pdf p_n. Let $Be(a_1,a_2) \equiv D_2(a_1:a_2)$ denote a beta distribution.

<u>Lemma 1.</u> Let ν be the invariant probability measure on $\mathfrak{O}(n)$. Let G be a random matrix defined on $\mathfrak{O}(n)$ such that $G(g) = g$ for $g \in \mathfrak{O}(n)$. Let Z be an $n\times 1$ random vector such that $\mathcal{L}(Z) = N_n(0,I)$. Then $\mathcal{L}(G_1) = \mathcal{L}(Z/\|Z\|)$ where $G_1 = (G_{11},\ldots,G_{n1})'$ is the first column of G.

<u>Proof:</u> Eaton (1972) p. 6.20 gives a construction for the unique invariant probability measure ν on $\mathfrak{O}(n)$. The result follows from the construction. $\|$

<u>Theorem 1.</u> Let $\mathcal{L}(X) \in S(n)$ such that $P(X = 0) = 0$. Then $\mathcal{L}(X/\|X\|) = \mathcal{L}(Z/\|Z\|)$ where $\mathcal{L}(Z) = N_n(0,I)$.

Proof: Let G be the random matrix in Lemma 1. We take G to be independent of X. For a Borel set B in R^n, we consider

$$(1.1) \quad P\left(\frac{GX}{\|X\|} \in B\right) = \int P\left(\frac{gX}{\|X\|} \in B \mid G = g\right) d\nu(g) = P\left(\frac{X}{\|X\|} \in B\right)$$

since X and G are independent and $\mathcal{L}(X) \in S(n)$ with $P(X = 0) = 0$. On the other hand

$$(1.2) \quad P\left(\frac{GX}{\|X\|} \in B\right) = \int P\left(\frac{Gx}{\|x\|} \in B \mid X = x \neq 0\right) dF(x)$$

$$= \int P_\nu\left(\frac{Gx}{\|x\|} \in B\right) dF(x) = P_\nu(G_1 \in B)$$

where F is the distribution function of X. Here the third equality holds because ν is left and right invariant so that we can replace x by $(\|x\|,0,\ldots,0)' \in R^n$. From (1.1), (1.2) and Lemma 1, the result follows. $\|$

As an alternative proof, we may argue as follows. Since $T(X) \equiv X/\|X\|$ satisfies $T(gX) = gT(X)$ for $g \in \mathcal{O}(n)$ and since $\mathcal{L}(X) \in S(n)$ with $P(X = 0) = 0$, $\mathcal{L}(T) \in S(n)$. From the uniqueness of invariant measures, T must have the uniform probability distribution on $\{x \mid \|x\| = 1\}$ since $\|T\| = 1$. Hence $\mathcal{L}(T) = \mathcal{L}(Z/\|Z\|)$ where $\mathcal{L}(Z) = N_n(0,I)$.

This theorem says that whatever $\mathcal{L}(X)$ may be so long as $\mathcal{L}(X) \in S(n)$ and $P(X = 0) = 0$, $\mathcal{L}(X/\|X\|)$ is equal to $\mathcal{L}(Z/\|Z\|)$ where $\mathcal{L}(Z) = N_n(0,I)$. In this sense, $X/\|X\|$ or functions of it are distribution-free.

Theorem 2. Let $\mathcal{L}(X) \in S(n)$ such that $P(X = 0) = 0$. Let A be an $n \times n$ symmetric matrix. Then $\mathcal{L}(X'AX/X'X) = \mathcal{L}(\sum_{i=1}^{n} d_j y_j)$ where $\mathcal{L}(y_1,\ldots,y_n) = D_n(\frac{1}{2},\ldots,\frac{1}{2}:\frac{1}{2})$ and d_j's are the latent roots of A.

In particular, if $A^2 = A$ and rank $(A) = k$, then $\mathcal{L}(X'AX/X'X) = Be(\frac{k}{2}, \frac{n-k}{2})$.

Proof: By Theorem 1, without loss of generality we can assume $\mathcal{L}(X) = N_n(0,I)$. Hence the result is immediate. $\|$

Press (1969) reviews $\mathcal{L}(X'AX/X'X)$ when $\mathcal{L}(X)$ is normal and Efron (1969) shows that $\mathcal{L}(\sqrt{n(n-1)}\ \bar{y}/[\sum_{i=1}^{n}(y_i-\bar{y})^2]^{\frac{1}{2}})$ does not depend on the normality of $\mathcal{L}(y_1,\ldots,y_n)$ but on the sphericity of Y where $\bar{y} = \sum_{i=1}^{n} y_i/n$.

As an application of Theorem 1, we give an example. Let $\mathcal{L}(u|v)$ denote the conditional distribution of a random vector u given a random vector v.

Example 1. Let u and v be $n\times 1$ random vectors such that $P(v=0) = 0$ and $\mathcal{L}(u|v) \in S(n)$ with $P(u = 0|v) = 0$. We consider the distribution of $w \equiv u'v/\|u\|\ \|v\|$. By Theorem 1 $\mathcal{L}(w|v)$ does not depend on $\mathcal{L}(u|v)$ and so we can assume $\mathcal{L}(u|v) = N_n(0,I)$ without loss of generality. Choose $g \in \mathfrak{G}(n)$ such that g has the first row $v'/\|v\|$ and let $y = gu$ $(y = (y_1,\ldots,y_n)')$. Then $w = y_1/\|y\|$ and $\mathcal{L}(y|v) = \mathcal{L}(u|v)$, which is assumed to be equal to $N_n(0,I)$. Define $t = \sqrt{n-1}\ y_1/(\sum_{i=2}^{n-1} y_i^2)^{\frac{1}{2}}$. Then $\mathcal{L}(t|v)$ is a t-distribution with degrees of freedom $n-1$, denoted by $t(n-1)$ and so it is independent of v. Since $w = \frac{t}{n-1}/[1 + t^2/n-1]^{\frac{1}{2}}$ is strictly increasing function of t, $\mathcal{L}(w|v)$ does not depend on the condition and $\mathcal{L}(w) = \mathcal{L}(w'v)$ is determined by $t(n-1)$. Thus $\sqrt{n-1}\ w/\sqrt{1-w^2} = t$ is distributed as $t(n-1)$ whatever $\mathcal{L}(v)$ may be so long as $P(v = 0) = 0$. In particular, it holds if u is independent of v or if v is a constant.

For example, consider the sample correlation coefficient

$$(1.3) \qquad r = \sum_{i=1}^{n}(u_i - \bar{u})(v_i - \bar{v})/[\sum_{i=1}^{n}(u_i - \bar{u})^2 \sum_{i=1}^{n}(v_i - \bar{v})^2]^{\frac{1}{2}}$$

where $u = (u_1,\ldots,u_n)'$, $v = (v_1,\ldots,v_n)'$ and $\bar{u} = \sum_{i=1}^{n} u_i/n$, $\bar{v} = \sum_{i=1}^{n} v_i/n$. Suppose u and v are independent, $P(v = 0) = 0$ and $\mathcal{L}(u) \in S(n)$

with $P(u = 0) = 0$. Let $M = e(e'e)^{-1}e'$ where $e = (1,\dots,1)' \in R^n$. Then $r = u'(I-M)v/[u'(I-M)u \cdot v'(I-M)v]^{\frac{1}{2}}$. Define $y = gu$ and $z = gv$ where $g \in \mathcal{O}(n)$ and $g'(I-M)g = \mathrm{diag}\{1,\dots,1,0\}$, where $\mathrm{diag}\{a_1,a_2,\dots,a_n\}$ denotes the diagonal matrix with diagonal element $a_1,\dots,a_n$. Let $\tilde{y}$ and $\tilde{z}$ be the vectors consisting of the first n-1 elements of y and z respectively. Then r can be written as $r = \tilde{y}'\tilde{z}/\|\tilde{y}\|\ \|\tilde{z}\|$. Since $\mathcal{L}(y) \in S(n)$ with $P(y = 0) = 0$, $\mathcal{L}(\tilde{y}) \in S(n-1)$ with $P(\tilde{y} = 0) = 0$. Therefore from the argument above, whatever $\mathcal{L}(v)$ or $\mathcal{L}(\tilde{z})$ may be so long as $P(v = 0) = 0$, $\mathcal{L}(r)$ does not depend on $\mathcal{L}(v)$ and $\mathcal{L}(\sqrt{n-2}\ r/\sqrt{1-r^2}) = t(n-2)$. That is, the distribution of r does not depend on either the normality of $\mathcal{L}(u)$ or the distribution of v if u and v are independent, $P(v = 0) = 0$ and $\mathcal{L}(u) \in S(n)$ with $P(u = 0) = 0$.

§ 2. UMP tests for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_1(\Sigma)$.

First we give an alternative form of L-S Theorem for an infinite group. Let $\mathfrak{X}$ be a set with a σ-field and a measure μ, and suppose $\mathcal{G}$ is a group acting bimeasurably on the left of $\mathfrak{X}$ by $x \to g\,x$. Let $\mathfrak{F}$ be the class of the pdf's invariant under $\mathcal{G}$. Let $t:\mathfrak{X} \to \mathfrak{T}$ be a maximal invariant where $\mathfrak{T}$ is the range of t.

Lemma 2. (L-S) Suppose that for a given pdf $h \notin \mathfrak{F}$, there exists a map s from $\mathfrak{T}$ into $\mathfrak{X}$ such that $h(s(t(x)))$ is integrable with respect to μ. Then the test ϕ defined by

$$(2.1)\qquad \phi(x) = \begin{cases} 1 & \text{if } h(x) > kh(s(t(x))) \\ \gamma(x) & \qquad\quad = \\ 0 & \qquad\quad < \end{cases}$$

is a MP test of level α for testing $\mathfrak{F}$ versus h provided

$$(2.2)\qquad \mathcal{E}_{f_0}\phi = \alpha \text{ and } \mathcal{E}_f\phi \le \alpha \text{ for all } f \in \mathfrak{F}$$

where $f_0(x) = I^{-1}h(s(t(x)))$ and $I = \int h(s(t(x)))d\mu$. k is a constant.

Proof: Since $f_0 \in \mathfrak{F}$ and $If_0(x) = h(s(t(x)))$, ϕ is a MP test of its level for testing f_0 versus h. Since any test ϕ^* satisfying $\mathcal{E}_f\phi^* \le \alpha$ for all $f \in \mathfrak{F}$ satisfies $\mathcal{E}_{f_0}\phi^* \le \alpha$, from (2.2) ϕ is a MP test of level α for testing $\mathfrak{F}$ versus h. ||

We implicitly assumed the measurability of t and s in Lemma 2. In spite of its general form, the condition on the existence of a map s and the condition (2.2) are rather restrictive. However if $\mathfrak{X} = R^n$ and $\mathcal{G}$ is a compact subgroup of the group of nonsingular matrices $G\ell(n)$, then Lemma 2 is rather easily applicable as in the case of the permutation tests where $\mathcal{G}$ is finite and, as will be seen, in our problem. We note that condition (2.2) can be replaced by a condition of similarity

$$(2.3) \qquad \mathcal{E}_f\phi = \alpha \quad \text{for all} \quad f \in \mathfrak{F}.$$

If $\mathfrak{X} = R^n$ and $\mathcal{G}$ is a compact subgroup of $G\ell(n)$, then (2.3) is implied by

$$(2.4) \qquad \int\phi(gx)d\nu(g) = \alpha$$

where ν is the invariant probability measure on $\mathcal{G}$. (See L-S(1949) for the case of a finite group $\mathcal{G}$.) In fact L-S(1949) proved that the class of tests with property (2.4) forms an essentially complete class.

For our problems stated in §0, $\mathfrak{X} = R^n$, $\mathcal{G} = \mathfrak{O}(n)$ and μ is a Lebesgue measure. Clearly a maximal invariant is $t(x) = \|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$.

Theorem 3. For a fixed $\Sigma\ (\neq cI,\ c > 0)$, the test ϕ defined by

$$(2.5) \qquad \phi(x) = \begin{cases} 1 & \text{if } x'\Sigma^{-1}x/x'x < k \\ 0 & \qquad\qquad\qquad \ge k \end{cases}$$

is a UMP test of its level for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_1(\Sigma)$. Here for a given level α, k can be determined by

$$(2.6)\qquad \int\cdots\int_A p_n(t_1,\ldots,t_{n-1})\prod_{i=1}^{n-1} dt_i = \alpha$$

where p_n is the pdf of $D_n(\frac{1}{2},\ldots,\frac{1}{2}:\frac{1}{2})$, $A = \{\sum_{j=1}^{n} d_j t_j < k\}$ with $t_n = 1 - \sum_{i=1}^{n-1} t_i$ and d_j's are the latent roots of Σ^{-1}.

Proof: The latter part is clear from Theorem 2. Given any $h_0 \in \mathfrak{F}_1(\Sigma)$, i.e., $h_0(x) = x|\Sigma|^{-\frac{1}{2}} q_0(x'\Sigma^{-1}x)$, we take as a function s in Lemma 2 $s(t) = \ell t\,\Sigma^{\frac{1}{2}} a$ where a is any $n\times 1$ vector such that $a'a = 1$ and ℓ is any real number $(\ell \neq 0)$. Then by Lemma 2 the test ϕ defined by (2.1) with $\gamma(x) = 0$ and $k = 1$ is a MP test of its level provided $\mathcal{E}_f\phi$ is constant for all $f \in \mathfrak{F}_0$, since $I = \int h_0(s(t(x)))dx = \int c|\Sigma|^{-\frac{1}{2}} q_0(\ell^2\|x\|^2)dx < \infty$. To be a bit more precise, let $\tilde{\mathfrak{F}}_1(\Sigma)$ be the class of the pdf's of the form (0.1) where in (0.1) q is strictly decreasing on $[0, \infty)$. Clearly $\tilde{\mathfrak{F}}_1(\Sigma) \subset \mathfrak{F}_1(\Sigma)$. If $h_0 \in \tilde{\mathfrak{F}}_1(\Sigma)$, i.e., q_0 is strictly decreasing, $\phi(x) = 1$ if and only if $x'\Sigma^{-1}x < \ell^2\|x\|^2$, which is nothing but ϕ in (2.5). Further $\mathcal{E}_f\phi$ is constant for all $f \in \mathfrak{F}_0$ from (2.6). Since ϕ does not depend on h_0, it is UMP of its level for $\tilde{\mathfrak{F}}_1(\Sigma)$. Now for $h_0 \in \mathfrak{F}_1(\Sigma)$, we take any $h_1 \in \tilde{\mathfrak{F}}_1(\Sigma)$ and define $h_m(x) = (1 - \frac{1}{m})h_0(x) + \frac{1}{m}h_1(x)$. Since $h_m \in \tilde{\mathfrak{F}}_1(\Sigma)$, the test ϕ in (2.5) dominates any test ψ under h_m, that is, $\int\phi h_m dx \geq \int\psi\, h_m dx$. Letting $m \to \infty$ and applying Scheffe's Lemma, $\int\phi h_0 dx \geq \int\psi h_0 dx$, which completes the proof. ||

As corollaries, we give examples for some structured Σ where the UMP tests do not depend on unknown parameters in Σ.

Example 2. $\Sigma = \sigma^2\Sigma_0$ (Σ_0: known).

From Theorem 3, the test with critical region (c.r.) $x'\Sigma_0^{-1}x/x'x < k$ is UMP for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_1(\sigma^2\Sigma_0)$.

Example 3. $\Sigma = \lambda_1(I-M) + \lambda_2 M$, $\lambda_1 > \lambda_2 > 0$, $M^2 = M$ (M: known). By Theorem 3, for a fixed $(\lambda_1,\lambda_2)(\lambda_1 > \lambda_2 > 0)$, the test with c.r. $x'\Sigma^{-1}x/x'x < k$ is UMP. But $x'\Sigma^{-1}x/x'x = \lambda_1^{-1} + (\lambda_2^{-1} - \lambda_1^{-1})x'Mx/x'x$. Therefore $x'Mx/x'x < k'$ is UMP. Since the test does not depend on the fixed (λ_1,λ_2), it is UMP for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_1(\Sigma)$ with $\lambda_1 > \lambda_2 > 0$. The cut-off point k' is determined from $\mathcal{L}(x'Mx/x'x) = Be(\frac{\ell}{2},\frac{n-\ell}{2})$ by Theorem 2 where $\ell = \operatorname{rank}(M)$. If $\lambda_2 > \lambda_1 > 0$, $x'Mx/x'x > k$ is UMP. As a special case of this type, we let $\lambda_1 = \sigma^2(1-\rho)$, $\lambda_2 = \sigma^2(1-\rho + n\rho)$ and $M = e(e'e)^{-1}e'$ $(e = (1,\ldots,1)' \in R^n)$. Then the test $(e'x)^2/x'x > k$ is UMP for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_1(\Sigma(\sigma^2,\rho))$ with $\rho > 0$, and the cut-off point is calculated from $Be(\frac{1}{2}, \frac{n-1}{2})$. We shall evaluate the power of this test. Let $g \in \mathfrak{O}(n)$ such that g has $e/\sqrt{n}$ as its first column and $g'\Sigma^{-1}g = \operatorname{diag}\{\tau_1,\tau_2,\ldots,\tau_2\}$ where $\tau_1 = [\sigma^2(1-\rho + n\rho)]^{-1}$ and $\tau_2 = [\sigma^2(1-\rho)]^{-1}$. Let $y = g'x$. Then $(e'x)^2/n\|x\|^2 = y_1^2/\|y\|^2$ and for $h(x)$ in $\mathfrak{F}_1(\Sigma)$ the pdf of y is given by $c(\tau_1\tau_2^{n-1})^{\frac{1}{2}}q(\tau_1 y_1^2 + \tau_2 \sum_{i=2}^{n} y_i^2)$ where $y = (y_1,\ldots,y_n)'$. Hence from Theorem 2,
$\mathcal{L}((\sqrt{\tau_1}\, y_1)^2/[(\sqrt{\tau_1}\, y_1)^2 + \sum_{i=2}^{n}(\sqrt{\tau_1}\, y_i)^2]) = Be(\frac{1}{2}, \frac{n-1}{2})$, so
$\mathcal{L}((n-1)\beta y_1^2/\sum_{i=2}^{n} y_i^2) = F(1, n-1) \equiv F$ distribution with degrees of freedom 1 and n-1 where $\beta = \tau_1/\tau_2$. Therefore $y_1^2/\sum_{i=2}^{n} y_i^2$ is distributed as $F(1, n-1)/(n-1)\beta$ and the power function is given by

$$\pi(\phi, \rho) = \int_\delta^\infty F(u{:}1, n-1)du \qquad 0 < \rho < 1$$

where $\delta = (1-\rho)[1-\rho + n\rho]^{-1}k(1-k)^{-1}(n-1)^{-1}$ and $F(u{:}1, n-1)$ is the pdf of $F(1{:}n-1)$. We note that the power function does not depend on h or q and σ^2.

Example 4. $\Sigma^{-1} = \lambda_1 I + \lambda_2 A$, $\lambda_1 > 0$ (A: known).
Here λ_2 takes the values for which Σ^{-1} is positive definite. As above,

$x'Ax/x'x < k$ is UMP when $\lambda_2 > 0$. If $\lambda_2 < 0$ the inequality is reversed. $\mathcal{L}(x'Ax/x'x)$ is expressed in Theorem 2. As a special case, we take Σ of the form

$$\Sigma^{-1} = \tau(1+\rho^2)I - \tau\rho \begin{bmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix} \quad (|\rho| < 1) .$$

This form is often treated in serial correlation problems. In this case $\sum_{i=2}^{n} x_i x_{i-1}/x'x > k$ is UMP for $\rho > 0$. This test coincides with the test under normality. (See Anderson (1948).)

§ 3. UMP test for testing $\mathfrak{F}_0$ versus $\mathfrak{F}_2(\mu)$.

Theorem 4. For testing $\mathfrak{F}_0$ versus $\mathfrak{F}_2(\mu)$ with $\mu > 0$, the test defined by

$$\phi(x) = \begin{cases} 1 & \text{if } a_0'x/\|a_0\|\,\|x\| > k \\ 0 & \qquad\qquad\qquad\quad \leq k \end{cases} \tag{3.1}$$

is a UMP test of its level. For a given level α, the cut-off point can be calculated by

$$\int_{k'}^{\infty} t(u:n-1)du = \alpha \quad (k' = \sqrt{n-1}\,k/(1-k^2)^{\frac{1}{2}}). \tag{3.2}$$

where $t(u:n-1)$ is the pdf of $t(n-1)$ distribution.

Proof: The latter part is clear from Example 1. We fix $\mu > 0$. Given any $h_0 \in \mathfrak{F}_2(\mu)$, i.e., $h_0(x) = cq_0(\|x-\mu a_0\|^2)$, we take as a function s in Lemma 2 $s(t) = tb$ where b is any $n\times 1$ vector such that $b'b = 1$. Then by Lemma 2, the test ϕ defined by (2.1) with $\gamma(x) = 0$ and $k = 1$ is a MP test of its level provided $\mathcal{E}_f\phi$ is constant for all $f \in \mathfrak{F}_0$. If q_0 is strictly decreasing, $\phi(x) = 1$ if and only if $\|x-\mu a_0\|^2 < \|x\|\,\|b-\mu a_0\|^2$ or $x'a_0/\|a_0\|\,\|x\| > b'a_0/\|a_0\|$. Let $k = b'a/\|a_0\|$

so ϕ is given by (3.1). Further from Example 1 $\mathcal{E}_f\phi$ is constant for all $f \in \mathfrak{F}_0$. Since ϕ does not depend on h_0 and μ, ϕ in (3.1) is UMP for $\mathfrak{F}_0$ versus $\mathfrak{F}_2(\mu)$ with q strictly increasing. But in the same way as in the proof of Theorem 3, it is easily shown to be UMP for $\mathfrak{F}_0$ versus $\mathfrak{F}_2(\mu)$. ||

We remark that in §2,3 we treated one-sided testing problems for unknown parameters (e.g., $\rho > 0$ or $\rho < 0$, $\mu > 0$ or $\mu < 0$). This leaves the two-sided testing problems. For example, such problems as testing $\mathfrak{F}_0$ versus $\mathfrak{F}_2(\mu)$ with $\mu \neq 0$ are left open although one may conjecture that the two-sided test $|a_0'x|/\|a_0\|\ \|x\| > k$ is UMP unbiased. Here we note that $(a_0'x, \|x\|)$ is a complete and sufficient statistic under $\mathfrak{F}_2(\mu)$. Hence if a UMP unbiased test exists, it is unique and it must coincide with the UMP unbiased $|a_0'x|/\|a_0\|\ \|x\| > k$ under the normal alternatives, which was treated in L-S(1949).

§ 4. Applications to linear models.

Let $y = X\beta + u$ be a regression model where $y{:}n\times 1$ and $X{:}n\times k$ with rank $(X) = k$. For the theory of the ordinary least squares (LS) estimation to be applicable, it is sufficient to asssume $\mathcal{E}(u) = 0$ and $\mathcal{E}(uu') = \sigma^2 I_n$ for the error term u. Customarily it is thought that to carry out a testing procedure the assumption of normality for u is needed (e.g., Scheffè (1959).) Assuming normality for u results in assuming the independence of u_i's. However here we simply assume $\mathcal{L}(u) \in S(n)$ with $P(u = 0) = 0$ and $\mathcal{E}\|u\|^2 < \infty$. This implies that $\mathcal{E}u = 0$ and $\mathcal{E}uu' = \sigma^2 I$ for $\sigma^2 > 0$. Consider the problem of testing a general linear hypothesis $A\beta = a$ where $A{:}r \times k$ with rank $(A) = r$. (See Scheffè (1959), Lehmann (1959) and Eaton (1972).) Without loss of generality we assume $a = 0$. This problem can be stated in a canonical form as follows. The model is

$$(4.1)\quad (\underset{k-r}{z_1'}, \underset{r}{z_2'}, \underset{n-k}{z_3'})' = (\gamma_1', \gamma_2', 0')' + (v_1', v_2', v_3')'$$

where $(z_1', z_2', z_3') \equiv z = Py$ for some $P \in \mathcal{O}(n)$ and γ_i's, v_i's have the corresponding orders of the z_i's. Since the sphericity of u is preserved under orthogonal transformations, $\mathcal{L}(v) \in S(n)$ and $P(v = 0) = 0$. The hypothesis is $H_0 : \gamma_2 = 0$. The problem is invariant under $(z_1', z_2', z_3') \to c(z_1'g_1' + b_1', z_2'g_2', z_3'g_3')$ where $g_1 \in \mathcal{O}(k-r)$, $g_2 \in \mathcal{O}(r)$, $g_3 \in \mathcal{O}(n-k)$, $b_1 \in R^{k-r}$ and $c > 0$. Then a maximal invariant is the usual F ratio $Q = (n-k)z_2'z_2/rz_3'z_3$ or $w = z_2'z_2/(z_2'z_2 + z_3'z_3)$. Hence by invariance we consider tests based upon Q or w. From Theorem 2, under H_0 $\mathcal{L}(Q) = F(r,n-k)$ or $\mathcal{L}(w) = Be(\frac{r}{2}, \frac{n-k}{2})$. It is noted that a marginal distribution of a spherically symmetric distribution is spherically symmetric. Therefore we may use the usual F-test. This fact can be interpreted another way. The usual F-test under normality is quite robust for nonnormality so long as $\mathcal{L}(u) \in S(n)$ and $P(u = 0) = 0$. It is noted that the existence of the pdf is not assumed and that the F-test is also robust for nonexistence of moments although there the LS theory may fail. The robustness of the F-test has been studied by Box and Watson (1962) when the error vector $u = (u_1,\ldots,u_n)'$ is a random sample from symmetric nonnormal distribution. The assumption of the independence of $u_1,\ldots,u_n$ with our assumption of spherical symmetry implies that $\mathcal{L}(u)$ is normal. Of course, in the situation treated by Box and Watson, the test statistic Q no longer has an F-distribution.

Further without normality, we can test such hypotheses as H_0: $\mathcal{E}(uu') \equiv \Sigma = \sigma^2 I$ versus H_1: $\Sigma = \sigma^2(1-\rho)I + \sigma^2 ee'$ $(\rho > 0)$ etc. (see Examples 3,4) under the assumption that the pdf f of u is in $\mathcal{F}_1(I)$, i.e., $f(u) = cq(u'u)$ where σ^2 is absorbed into q. In the above case, in terms of the canonical form, we test H_0:$P \Sigma P' = \sigma^2 I$ versus H_1:$P \Sigma P' = \sigma^2(1-\rho)I + \sigma^2\rho aa'(\rho > 0)$ where $a = Pe$. In this case the structure of X affects the properties of the test. (See Anderson (1948).) However applying invariance, from Example 3, we can always obtain a UMPI (invariant) test with critical region $(z_3'a_3)^2/z_3'z_3 < k$ provided $a_3 \neq 0$ where $a = (a_1', a_2', a_3')'$. Further the cut-off point and the power function can be calculated as in Examples 3.

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